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AUTHOR(S):

川崎, 敏治; 小林, 鉄雄

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# Existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces and some examples

日本大学 川崎敏治 (toshiharu.kawasaki@nifty.ne.jp)  
(Toshiharu Kawasaki, Nihon University)  
日本大学 小林鉄雄 (tekoba@ge.ce.nihon-u.ac.jp)  
(Tetsuo Kobayashi, Nihon University)

## Abstract

In this paper we prove a fixed point theorem and mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space. Moreover we give some examples of widely more generalized hybrid non-self mapping.

## 1 Introduction

Let  $H$  be a real Hilbert space and let  $C$  be a non-empty subset of  $H$ . In 2010, Kocourek, Takahashi and Yao [14] defined a class of nonlinear mappings in a Hilbert space. A mapping  $T$  from  $C$  into  $H$  is said to be generalized hybrid if there exist real numbers  $\alpha$  and  $\beta$  such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any  $x, y \in C$ . We call such a mapping an  $(\alpha, \beta)$ -generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an  $(\alpha, \beta)$ -generalized hybrid mapping is nonexpansive [19] for  $\alpha = 1$  and  $\beta = 0$ , that is,  $\|Tx - Ty\| \leq \|x - y\|$  for any  $x, y \in C$ . It is nonspreading [16] for  $\alpha = 2$  and  $\beta = 1$ , that is,  $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$  for any  $x, y \in C$ . It is also hybrid [20] for  $\alpha = \frac{3}{2}$  and  $\beta = \frac{1}{2}$ , that is,  $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$  for any  $x, y \in C$ . They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [15] and Iemoto and Takahashi [9]. Moreover they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping  $T$  from  $C$  into  $H$  is said to be super hybrid if there exist real numbers  $\alpha, \beta$  and  $\gamma$  such that

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + (1 - \alpha + \gamma)\|x - Ty\|^2 \\ & \leq (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\ & \quad + (\alpha - \beta)\gamma\|x - Tx\|^2 + \gamma\|y - Ty\|^2 \end{aligned}$$

for any  $x, y \in C$ . A generalized hybrid mapping with a fixed point is quasinonexpansive. However, a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the authors [11] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [14]. A mapping  $T$  from  $C$  into  $H$  is said to be widely generalized hybrid if there exist real numbers  $\alpha, \beta, \gamma, \delta, \varepsilon$  and  $\zeta$  such that

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ & + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0 \end{aligned}$$

for any  $x, y \in C$ . Moreover the authors [12] defined a class of nonlinear mappings in a Hilbert space which covers the class of super hybrid mappings and the class of widely generalized hybrid mappings. A mapping  $T$  from  $C$  into  $H$  is said to be widely more generalized hybrid if there exist real numbers  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$  and  $\eta$  such that

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ & + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for any  $x, y \in C$ . Then we prove fixed point theorems for such new mappings in a Hilbert space. Moreover we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [14] for super hybrid mappings. On the other hand, Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping  $T$  from  $C$  into  $H$  is said to be extended hybrid if there exist real numbers  $\alpha, \beta$  and  $\gamma$  such that

$$\begin{aligned} & \alpha(1 + \gamma)\|Tx - Ty\|^2 + (1 - \alpha(1 + \gamma))\|x - Ty\|^2 \\ & \leq (\beta + \alpha\gamma)\|Tx - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma\|x - Tx\|^2 - \gamma\|y - Ty\|^2 \end{aligned}$$

for any  $x, y \in C$ . Moreover they proved a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping.

In this paper, using an idea of [8], we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Moreover we prove mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

## 2 Preliminaries

Throughout this paper, we denote by  $\mathbb{N}$  the set of positive integers and by  $\mathbb{R}$  the set of real numbers. Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . We

denote the strong convergence and the weak convergence of  $\{x_n\}$  to  $x \in H$  by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$ , respectively. Let  $A$  be a non-empty subset of  $H$ . We denote by  $\overline{\text{co}}A$  the closure of the convex hull of  $A$ . In a Hilbert space, it is known that

$$\|(1 - \lambda)x + \lambda y\|^2 = (1 - \lambda)\|x\|^2 + \lambda\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2 \quad (2.1)$$

for any  $x, y \in H$  and for any  $\lambda \in \mathbb{R}$ ; see [19]. Moreover in a Hilbert space, we obtain that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \quad (2.2)$$

for any  $x, y, z, w \in H$ . Let  $C$  be a non-empty subset of  $H$  and let  $T$  be a mapping from  $C$  into  $H$ . We denote by  $F(T)$  the set of fixed points of  $T$ . A mapping  $T$  from  $C$  into  $H$  with  $F(T) \neq \emptyset$  is said to be quasi-nonexpansive if  $\|x - Ty\| \leq \|x - y\|$  for any  $x \in F(T)$  and for any  $y \in C$ . It is well-known that the set  $F(T)$  of fixed points of a quasi-nonexpansive mapping  $T$  is closed and convex; see Ito and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space; see, for instance, [22]. Let  $C$  be a non-empty closed convex subset of  $H$  and  $x \in H$ . Then, we know that there exists a unique nearest point  $z \in C$  such that  $\|x - z\| = \inf_{y \in C} \|x - y\|$ . We denote such a correspondence by  $z = P_C x$ . The mapping  $P_C$  is said to be the metric projection from  $H$  onto  $C$ . It is known that  $P_C$  is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for any  $x \in H$  and for any  $u \in C$ ; see [19] for more details. For proving a mean convergence theorem, we also need the following lemma proved by Takahashi and Toyoda [21].

**Lemma 2.1.** *Let  $C$  be a non-empty closed convex subset of  $H$ . Let  $P_C$  be the metric projection from  $H$  onto  $C$ . Let  $\{u_n\}$  be a sequence in  $H$ . If  $\|u_{n+1} - u\| \leq \|u_n - u\|$  for any  $u \in C$  and for any  $n \in \mathbb{N}$ , then  $\{P_C u_n\}$  converges strongly to some  $u_0 \in C$ .*

Let  $\ell^\infty$  be the Banach space of bounded sequences with supremum norm. Let  $\mu$  be an element of  $(\ell^\infty)^*$  (the dual space of  $\ell^\infty$ ). Then, we denote by  $\mu(f)$  the value of  $\mu$  at  $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$ . Sometimes, we denote by  $\mu_n(x_n)$  the value  $\mu(f)$ . A linear functional  $\mu$  on  $\ell^\infty$  is said to be a mean if  $\mu(e) = \|\mu\| = 1$ , where  $e = (1, 1, 1, \dots)$ . A mean  $\mu$  is said to be a Banach limit on  $\ell^\infty$  if  $\mu_n(x_{n+1}) = \mu_n(x_n)$ . We know that there exists a Banach limit on  $\ell^\infty$ . If  $\mu$  is a Banach limit on  $\ell^\infty$ , then for  $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$ ,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if  $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$  and  $x_n \rightarrow a \in \mathbb{R}$ , then we obtain  $\mu(f) = \mu_n(x_n) = a$ . See [18] for the proof of existence of a Banach limit and its other elementary properties. By means and the Riesz theorem, we have the following result; see [17] and [18].

**Lemma 2.2.** *Let  $H$  be a Hilbert space, let  $\{x_n\}$  be a bounded sequence in  $H$  and let  $\mu$  be a mean on  $\ell^\infty$ . Then there exists a unique point  $z_0 \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$  such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle$$

for any  $y \in H$ .

Kawasaki and Takahashi [12] proved by Lemma 2.2 the following fixed point theorem.

**Theorem 2.1.** *Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into itself which satisfies the following condition (1) or (2):*

- (1)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \gamma + \varepsilon + \eta > 0$  and  $\zeta + \eta \geq 0$ ;
- (2)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \beta + \zeta + \eta > 0$  and  $\varepsilon + \eta \geq 0$ .

*Then  $T$  has a fixed point if and only if there exists  $z \in C$  such that  $\{T^n z \mid n = 0, 1, \dots\}$  is bounded. In particular, a fixed point of  $T$  is unique in the case of  $\alpha + \beta + \gamma + \delta > 0$  on the conditions (1) and (2).*

As a direct consequence of Theorem 2.1, we obtain the following.

**Theorem 2.2.** *Let  $H$  be a real Hilbert space, let  $C$  be a bounded closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into itself which satisfies the following condition (1) or (2):*

- (1)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \gamma + \varepsilon + \eta > 0$  and  $\zeta + \eta \geq 0$ ;
- (2)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \beta + \zeta + \eta > 0$  and  $\varepsilon + \eta \geq 0$ .

*Then  $T$  has a fixed point. In particular, a fixed point of  $T$  is unique in the case of  $\alpha + \beta + \gamma + \delta > 0$  on the conditions (1) and (2).*

### 3 Fixed point theorem

Let  $H$  be a real Hilbert space and let  $C$  be a non-empty subset of  $H$ . A mapping  $T$  from  $C$  into  $H$  was said to be widely more generalized hybrid if there exist  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$  such that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned} \quad (3.1)$$

for any  $x, y \in C$ ; see Introduction. Such a mapping  $T$  is said to be  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid; see [12]. An  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [14] if  $\alpha + \beta = -\gamma - \delta = 1$  and  $\varepsilon = \zeta = \eta = 0$ . Moreover it is an extension of widely generalized hybrid mappings in the sense of Kawasaki and Takahashi [11]. By Theorem 2.2 we prove a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space.

**Theorem 3.1.** *Let  $H$  be a real Hilbert space, let  $C$  be a non-empty bounded closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which satisfies the following condition (1) or (2):*

- (1)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \gamma + \varepsilon + \eta > 0$ , and there exists  $\lambda \in \mathbb{R}$  such that  $\lambda \neq 1$  and  $(\alpha + \beta)\lambda + \zeta + \eta \geq 0$ ;
- (2)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \beta + \zeta + \eta > 0$ , and there exists  $\lambda \in \mathbb{R}$  such that  $\lambda \neq 1$  and  $(\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0$ .

Suppose that for any  $x \in C$ , there exist  $m \in \mathbb{R}$  and  $y \in C$  such that  $0 \leq (1 - \lambda)m \leq 1$  and  $Tx = x + m(y - x)$ . Then  $T$  has a fixed point. In particular, a fixed point of  $T$  is unique in the case of  $\alpha + \beta + \gamma + \delta > 0$  on the conditions (1) and (2).

*Example 3.1.* Let  $H = \mathbb{R}$ , let  $C = [0, \frac{\pi}{2}]$ , let  $Tx = (1 + 2x)\cos x - 2x^2$  and let  $\alpha = 1$ ,  $\beta = \gamma = 11$ ,  $\delta = -22$ ,  $\varepsilon = \zeta = -12$  and  $\eta = 1$ . Then  $T$  is an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$ ,  $\alpha + \beta + \gamma + \delta = 1 \geq 0$  and  $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ . Let  $\lambda = \frac{2+3\pi}{3(1+\pi)}$  and let  $m = 1 + \pi$ . Then  $0 \leq (1 - \lambda)m = \frac{1}{3} < 1$  and  $(\alpha + \beta)\lambda + \zeta + \eta = \frac{\pi-3}{1+\pi} \geq 0$ . Let  $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$  for any  $x \in C$ . Then  $Tx = x + m(y - x)$  and  $y \in C$ . Therefore by Theorem 3.1  $T$  has a unique fixed point.

## 4 Nonlinear ergodic theorems

In this section, using the technique developed by Takahashi [17], we prove mean convergence theorems of Baillon's type in a Hilbert space. Before proving the results, we need the following lemmas.

**Lemma 4.1.** Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which has a fixed point and satisfies the condition:

$$\alpha + \gamma + \varepsilon + \eta > 0, \text{ or } \alpha + \beta + \zeta + \eta > 0.$$

Then  $F(T)$  is closed.

**Lemma 4.2.** Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which has a fixed point and satisfies the condition (1) or (2):

- (1)  $\alpha + \beta + \gamma + \delta \geq 0$  and  $\alpha + \gamma + \varepsilon + \eta > 0$ ;
- (2)  $\alpha + \beta + \gamma + \delta \geq 0$  and  $\alpha + \beta + \zeta + \eta > 0$ .

Then  $F(T)$  is convex.

**Lemma 4.3.** Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which has a fixed point and satisfies the condition (1) or (2):

- (1)  $\alpha + \beta + \gamma + \delta \geq 0$ ,  $\alpha + \gamma > 0$  and  $\varepsilon + \eta \geq 0$ ;

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0 \text{ and } \zeta + \eta \geq 0.$$

Then  $T$  is quasi-nonexpansive.

Moreover we obtain the following.

**Lemma 4.4.** *Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta.$$

Then  $(1 - \lambda)T + \lambda I$  is quasi-nonexpansive.

Now we first obtain the following mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

**Theorem 4.1.** *Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0 \text{ and } \varepsilon + \eta \geq 0;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0 \text{ and } \zeta + \eta \geq 0.$$

Then for any  $x \in C(T; 0) = \{z \mid T^n z \in C \text{ for any } n \in \mathbb{N} \cup \{0\}\}$ ,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point  $p$  of  $T$ , where  $P$  is the metric projection from  $H$  onto  $F(T)$  and  $p = \lim_{n \rightarrow \infty} P T^n x$ .

Moreover we obtain the following.

**Theorem 4.2.** *Let  $H$  be a real Hilbert space, let  $C$  be a non-empty closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta.$$

Then for any  $x \in C(T; \lambda) = \{z \mid ((1 - \lambda)T + \lambda I)^n z \in C \text{ for any } n \in \mathbb{N} \cup \{0\}\}$ ,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point  $p$  of  $T$ , where  $P$  is the metric projection from  $H$  onto  $F(T)$  and  $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$ .

**Theorem 4.3.** Let  $H$  be a real Hilbert space, let  $C$  be a non-empty bounded closed convex subset of  $H$  and let  $T$  be an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$  which satisfies the following condition (1) or (2):

- (1)  $\alpha + \beta + \gamma + \delta \geq 0$ , and there exists  $\lambda \in \mathbb{R}$  such that  $0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta$ ;
- (2)  $\alpha + \beta + \gamma + \delta \geq 0$ , and there exists  $\lambda \in \mathbb{R}$  such that  $0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta$ .

Suppose that for any  $x \in C$ , there exist  $m \in \mathbb{R}$  and  $y \in C$  such that  $0 \leq (1 - \lambda)m \leq 1$  and  $Tx = x + m(y - x)$ . Then for any  $x \in C$ ,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point  $p$  of  $T$ , where  $P$  is the metric projection from  $H$  onto  $F(T)$  and  $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$ .

*Example 4.1.* Let  $H = \mathbb{R}$ , let  $C = [0, \frac{\pi}{2}]$ , let  $Tx = (1 + 2x) \cos x - 2x^2$  and let  $\alpha = 1$ ,  $\beta = \gamma = 11$ ,  $\delta = -22$ ,  $\varepsilon = \zeta = -12$  and  $\eta = 1$ . Then  $T$  is an  $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from  $C$  into  $H$ ,  $\alpha + \beta + \gamma + \delta = 1 \geq 0$  and  $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ . Let  $\lambda = \frac{2+3\pi}{3(1+\pi)}$  and  $m = 1 + \pi$ . Then  $0 \leq (1 - \lambda)m = \frac{1}{3} < 1$  and  $0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta = \frac{\pi-3}{1+\pi} < 1 = \alpha + \gamma + \varepsilon + \eta$ . Let  $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$  for any  $x \in C$ . Then  $Tx = x + m(y - x)$  and  $y \in C$ . Therefore by Theorem 4.3 for any  $x \in C$ ,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point  $p$  of  $T$ , where  $P$  is the metric projection from  $H$  onto  $F(T)$  and  $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$ .

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